

Noise Kernel for Self-similar Tolman Bondi Metric: Fluctuations on Cauchy Horizon

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We attempt to calculate the point separated Noise Kernel for self similar Tolman Bondi metric, using a method similar to that developed by Hu et. al for ultrastatic metrics referring to work by Page. In case of formation of a naked singularity, the noise kernel thus obtained is found to be regular except on the Cauchy horizon, where it diverges. The behaviour of the noise in case of the formation of a covered singularity is found to be regular.

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I. INTRODUCTION

In semiclassical theory of gravity [1, 2], the quantum expectation of the energy momentum tensor is central to studying quantum fields on spacetimes. Stochastic gravity is one step ahead of this [3, 4]. It addresses the effect of including fluctuations of the energy momentum tensor and its backreaction on the metric. This is accomplished using the Einstein Langevin Equation.

The central object in the calculations of stochastic gravity is the noise kernel which gives an idea of the stochastic source, that being in addition to the quantum stress tensor.

The noise kernel is the vacuum expectation value of symmetrized stress-energy bitensor for a quantum field in curved spacetime [6, 7]. It characterizes fluctuations of stress tensor, studies of which play a significant role in analysis of quantum effects in curved spacetime. These fluctuations lead to *induced* or *passive* fluctuations of the metric [3]. Backreaction problems in gravity and cosmology have been addressed [8] using Einstein-Langevin equations with noise described by the stochastic term. Induced fluctuations play an important role in such studies.

The challenge in this approach is to come up with a method to calculate the Noise Kernel for specific background metrics. There are various approaches that have been used for such calculations, in the case of Minkowski, de-Sitter, anti-deSitter and Schwarzschild Spacetimes in the coincidence limit [9–14].

Recently, Hu et. al. [15] have devised a method to compute Noise Kernel for quantum fields in Schwarzschild spacetime under the Gaussian approximation. This is based on method developed by Page [16]. Here they have used an approach to do computations by conformally rescaling the metric to the form of an ultrastatic (optical) metric. First the Noise Kernel is evaluated in this ultrastatic spacetime. Then the Noise Kernel of Schwarzschild spacetime is obtained by just rescaling back the result in the optical case.

In this paper, we employ a similar method and compute the Noise kernel for self similar Tolman Bondi metric under Gaussian approximation. This spacetime has been studied previously [5] and it has been demonstrated that stress energy tensor diverges at the Cauchy horizon, despite Cauchy horizon being perfectly regular. The motivation of calculating Noise Kernel in this background is to test the validity of the above mentioned result in view of backreaction.

Our method corresponds to the direct method as given by Hu et.al in which one can exploit spherical symmetry of the metric as against the partially covariant expansion method in their work. We obtain the Noise Kernel in this case which diverges at the Cauchy horizon, seemingly making backreaction important for consideration and hence questioning any attempt to study the spacetime in a semi-classical or perturbative manner.

In section 2 we introduce the building blocks of stochastic gravity, the noise kernel and the required Wightman function.

In section 3 we describe the conformal rescaling of the Tolman Bondi metric and demonstrate its conformally equivalent ultrastatic form. We obtain the expansions of Synge function and Van Vleck Morette determinant in coordinates of the ultrastatic metric. We then have a section containing an attempt to relate the Noise Kernel of this metric to that of self-similar Tolman Bondi. We argue here for a plausible divergence. In the end we summarize the results and have a discussion of interpretation of the divergence.

II. EINSTEIN LANGEVIN EQUATION, NOISE KERNEL AND WIGHTMAN FUNCTION IN OPTICAL METRIC

The Noise Kernel for quantized matter field is given by

$$N_{abc'd'}(x, x') = \frac{1}{2} \langle \{ \hat{t}_{ab}(x), \hat{t}_{c'd'}(x') \} \rangle \quad (1)$$

where

$$\hat{t}_{ab}(x) \equiv \hat{T}_{ab}(x) - \langle \hat{T}_{ab}(x) \rangle$$

$\langle \dots \rangle$ is the quantum expectation value taken with respect to a normalized state. \hat{T}_{ab} is the stress tensor operator of the field. The classical stress tensor for a scalar field ϕ is given by

$$T_{ab} = \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} \nabla^c \phi \nabla_c \phi + \frac{1}{6} (g_{ab} \square - \nabla_a \nabla_b + G_{ab}) \phi^2 \quad (2)$$

As fit for semi-classical approach the field ϕ is raised to the level of an operator whereas $g_{\mu\nu}$ is to be treated classically.

The noise kernel embodies the contributions of the higher correlation functions in the quantum field on account of which it may be used to interpret issues related to the quantum nature of spacetime. Two point functions of energy

momentum tensor would involve fourth order correlations of the quantum field for example which would affect the coherence of the geometry, if we were to employ the Einstein Langevin equation

$$G_{\mu\nu} = \langle \hat{T}_{\mu\nu} \rangle + \hat{\xi}_{\mu\nu}. \quad (3)$$

$\hat{\xi}_{\mu\nu}$ is a random variable (tensor) which is closely related to the noise kernel. The relations

$$\langle \hat{\xi}_{\mu\nu}(x) \rangle_s = 0 \quad \& \quad \langle \hat{\xi}_{\alpha\beta}(x) \hat{\xi}_{\mu'\nu'}(x') \rangle_s = N_{\alpha\beta\mu'\nu'}(x, x')$$

which completely characterizes $\hat{\xi}_{\alpha\beta}(x)$.

The statistical expectation

$$\langle \quad \rangle_s$$

is taken over various stochastic realizations of the Gaussian source $\hat{\xi}_{\mu\nu}(x)$. Each of the realizations would lead to a metric solution

$$h_{\alpha\beta}(x) = h_{\alpha\beta}^{(0)}(x) + 8\pi \int d^4y' \sqrt{-g(y')} G_{\alpha\beta\gamma'\delta'}^{(ret)}(x, y') \xi^{\gamma'\delta'}(y') \quad (4)$$

where $h_{\alpha\beta}^{(0)}$ is the solution to the semiclassical equation

$$G_{\mu\nu} = \langle \hat{T}_{\mu\nu} \rangle$$

The stochastic realization would contain information about coherence of geometry . The statistical covariance of the metric solutions would thus be related to the statistical covariance of $\hat{\xi}^{\mu\nu}(x)$ over its stochastic realizations.

The properties and the expressions of such a noise kernel, given a Gaussian state of quantum matter field, are as given in [17], where the Wightman function is defined as

$$G^+(x, x') = \langle \phi(x) \phi(x') \rangle$$

We give a short review of the method used in [15] for ultrastatic metric. The relation between the Noise Kernel and Wightman function is as given in the above mentioned reference.

The metric in a static spacetime takes the form

$$ds^2 = g_{\tau\tau}(\vec{x}) d\tau^2 + g_{ij}(\vec{x}) dx^i dx^j \quad (5)$$

This can be transformed into an ultra-static one, called the optical metric, by a conformal transformation. This optical metric takes the following form

$$ds^2 = dt^2 + g_{ij}(\vec{x}) dx^i dx^j \quad (6)$$

where the metric functions g_{ij} are independent of time t . The Synge function can thus take the form

$$\sigma(x, x') = \frac{1}{2}((t - t')^2 - \mathbf{r}^2)$$

where \mathbf{r}^2 is twice the spatial part of the Synge function and it depends only on spatial co-ordinates as in [15].

The Wightman function for an optical metric takes the form [15]

$$G^+(\delta t, \vec{x}, \vec{x}') = \frac{\kappa \sinh \kappa \mathbf{r}}{8\pi^2 \mathbf{r} [\cosh(\kappa \mathbf{r}) - \cosh(\kappa \delta t)]} U(\delta t, \vec{x}, \vec{x}') \quad (7)$$

where

$$\mathbf{T} = \frac{\kappa}{2\pi}$$

is the temperature of the thermal state considered¹

$$\delta t = t - t'$$

$$U(x, x') \equiv \Delta^{1/2}(x, x')$$

$$\Delta(x, x') \equiv \frac{1}{\sqrt{-g(x)}\sqrt{-g(x')}} \det(\sigma_{;ab'})$$

where $\Delta(x, x')$ is the Van Vleck Morette determinant.

III. CONFORMALLY OPTICAL FORM OF SELF SIMILAR TOLMAN BONDI METRIC

The self-similar Tolman Bondi metric is given by

$$ds^2 = dt^2 - R' dr^2 - R^2 d\theta^2 - R^2 \sin^2 \theta d\phi^2 \quad (8)$$

where $R(t, r)$ is the area radius

$$R^{3/2}(t, r) = r^{3/2} \left(1 - \frac{3}{2} \frac{t}{r} \sqrt{\lambda}\right) \quad (9)$$

let $z = t/r$, then

$$R^{3/2}(z) = r^{3/2} \left(1 - \frac{3}{2} z \sqrt{\lambda}\right) \quad (10)$$

$$R'(z) = \frac{1 - \frac{3}{2} z \sqrt{\lambda}}{(1 - \frac{3}{2} z \sqrt{\lambda})^{1/3}} \quad (11)$$

$$\dot{R} = -r \sqrt{\frac{\lambda}{R}} \quad (12)$$

The initial data for this metric is regular and a curvature singularity gets eventually formed at $r = 0$. It is naked if $\lambda < 1/8$ and is covered otherwise.

The self similar TB metric can be put in the form

$$ds^2 = \left(1 - \frac{R'^2}{z^2}\right) t^2 [dT^2 - f1[z]^2 dz^2 - z^2 f2[z]^2 d\Omega^2] \quad (13)$$

where

$$f1[z]^2 = \left(\frac{R'}{z^2 - R'^2}\right)^2, \quad f2[z]^2 = \frac{(1 - \frac{3}{2} z \sqrt{\lambda})^{4/3}}{z^2 (z^2 - R'^2)}$$

and

$$dT = \frac{dt}{t} + \frac{R'^2}{z(z^2 - R'^2)} dz$$

For the metric (13) the Synge function can be obtained in the following form as is done in [15] for Schwarzschild metric in optical form. We apply the same procedure for TB metric in optical form. The Synge function expansion used here is

$$\sigma = \sum_{ijk} s_{ijk}(z) \delta T^{2i} \eta^j \delta z^k$$

¹ In Schwarzschild spacetime, the κ has been chosen to be the surface gravity on the event horizon null surface. However, one can work with a general thermal state without ascribing this significance to it. Clearly, thermal states exist in any spacetime including Tolman Bondi metrics.

where $\delta T = (T - T')$, $\delta z = (z - z')$, $\eta + 1 = \cos(\theta)\cos(\theta') + \sin(\theta)\sin(\theta')\cos(\phi - \phi')$. The expression we obtain is as follows:

$$\begin{aligned}\sigma(x, x') = & \delta T^2/2 - (1/2)\delta z^2 f1[z]^2 + \eta z^2 f2[z]^2 + \\ & \delta z \eta (-z f2[z]^2 - z^2 f2[z] f2'[z]) + \\ & 1/2 \delta z^3 f1[z] f1'[z] + O[(x - x')^4]\end{aligned}\quad (14)$$

the function $U(x, x')$ can be expanded in powers of $(x - x')$ as follows:

$$\begin{aligned}U(x, x') = & 1 + \delta z^2 \left(\frac{f1'[z]}{6zf1[z]} - \frac{f2'[z]}{3zf2[z]} + \frac{f1'[z]f2'[z]}{6f1[z]f2[z]} \right) \\ & \delta z^2 \left(-\frac{f2''[z]}{6f2[z]} \right) + \eta \left(-\frac{1}{6} + \frac{f2[z]^2}{6f1[z]^2} - \frac{zf2[z]^2 f1'[z]}{6f1[z]^3} + \frac{2zf2[z]f2'[z]}{3f1[z]^2} \right) \\ & \eta \left(-\frac{z^2 f2[z]f1'[z]f2'[z]}{6f1[z]^3} + \frac{z^2 f2'[z]^2}{6f1[z]^2} + \frac{z^2 f2[z]f2''[z]}{6f1[z]^2} \right) \\ & + O[(x - x')^3]\end{aligned}\quad (15)$$

IV. NOISE KERNEL EXPRESSION FOR (SELF SIMILAR) TOLMAN BONDII SPACETIME

The Wightmann function in the Gaussian approximation in ultrastatic spacetime is given by [15]

$$G^+(x, x') = \frac{1}{8\pi} \left[\frac{1}{\sigma} + \frac{\kappa^2}{6} - \frac{\kappa^4}{180} (2\delta T^2 + \sigma) + O[(x - x')^4] \right] U(x, x') \quad (16)$$

The noise kernel, when points are separated can now be computed using the above Green's function, in usual way as given in [15]. Here we will present one component of the noise kernel for demonstrating some important results for our metric.

The expression for noise kernel(component given by $N_{TTT'T'}$ for points separated in T , where $\delta z = \eta = 0$ and $\delta T \neq 0$. The noise Kernel $N_{TTT'T'}$ can be expanded in terms of

coefficient of κ^0

$$\begin{aligned}= & \frac{z^4 f1[z]^2}{R'[z]^2 t^2 (t - \delta t)^2} \left[\frac{13}{12\delta T^8 \pi^4} + \right. \\ & (-4f1[z]^4 + 3z^2 f2[z]^2 f1'[z]^2 + 2zf1[z]f2[z] \\ & (-2zf1'[z]f2'[z] + f2[z](-2f1'[z] + zf1''[z])) + 4f1[z]^2(2f2[z]^2 + \\ & 2z^2 f2'[z]^2 + zf2[z](6f2'[z] + zf2''[z])))/(72\delta T^6 \pi^4 z^2 f1[z]^4 f2[z]^2) \\ & + (27f1[z]^7 + 50zf1[z]^4 f2[z]f1'[z](f2[z] + zf2'[z]) + \\ & 360z^3 f2[z]^3 f1'[z]^3(f2[z] + zf2'[z]) - 10f1[z]^5(3f2[z]^2 + 3z^2 f2'[z]^2 + zf2[z](16f2'[z] + \\ & 5zf2''[z])) - 5z^2 f1[z]f2[z]^2 f1'[z](z^2 f1'[z]f2'[z]^2 + f2[z]^2(f1'[z] + \\ & 48zf1''[z]) + 2zf2[z](24zf2'[z]f1''[z] + f1'[z](73f2'[z] + 36zf2''[z])) + \\ & 2zf1[z]^2 f2[z](-49z^3 f1'[z]f2'[z]^3 + z^2 f2[z]f2'[z](12zf2'[z]f1''[z] - f1'[z](209f2'[z] + \\ & 31zf2''[z])) + f2[z]^3(-49f1'[z] + 12z(f1''[z] + zf1^{(3)}[z])) \\ & + zf2[z]^2(12z(4zf1''[z]f2''[z] + f2'[z](10f1''[z] + zf1^{(3)}[z])) + \\ & f1'[z](-209f2'[z] + z(185f2''[z] + 72zf2^{(3)}[z])) + f1[z]^3(3f2[z]^4 + \\ & 3z^4 f2'[z]^4 + 2z^3 f2[z]f2'[z]^2(104f2'[z] + 49zf2''[z]) + \\ & z^2 f2[z]^2(678f2'[z]^2 + 67z^2 f2''[z]^2 - 8zf2'[z](-49f2''[z] + 3zf2^{(3)}[z])) + \\ & 2zf2[z]^3(104f2'[z] + z(13f2''[z] - 12z(5f2^{(3)}[z] + \\ & zf2^{(4)}[z])))))/(25920\delta T^4 \pi^4 z^4 f1[z]^7 f2[z]^4)]\end{aligned}\quad (17)$$

Coefficient of κ^2

$$= \frac{z^4 f1[z]^2}{R'[z]^2 t^2 (t - \delta t)^2} \left[\frac{5}{72 \delta T^6 \pi^4} + \right. \\ \left. (2f1[z]^4 + 9z^2 f2[z]^2 f1'[z]^2 + 2zf1[z]f2[z](8zf1'[z]f2'[z] + \right. \\ \left. f2[z](8f1'[z] + 3zf1''[z])) + 2f1[z]^2(5f2[z]^2 + \right. \\ \left. 5z^2 f2'[z]^2 - 2zf2[z](3f2'[z] + 4zf2''[z])) \right) / (2592 \delta T^4 \pi^4 z^2 f1[z]^4 f2[z]^2) \right] \quad (18)$$

Coefficient of κ^4

$$= \frac{z^4 f1[z]^2}{R'[z]^2 t^2 (t - \delta t)^2} \left[-\frac{53}{1080 \delta T^4 \pi^4} \right] + O[(x - x')^3] \quad (19)$$

where t depends on T and z with $\delta t = t\delta T + O(\delta T)^2$. The above expression for the Noise Kernel has been obtained after conformal transformation of the optical form back to the original form (13).

We have displayed the expression of $N_{TTT'T'}$ for points separated in T co-ordinate only. This turns out to be interesting as it yields a divergence at the Cauchy horizon with leading orders in separation. This takes place despite the Cauchy horizon itself being regular in that there is no curvature singularity along it.

In case of the naked singular solution, the Cauchy Horizon is given by the smaller root z_- of $(z^2 - R'(z)^2)$ for self similar TB metric [5]. This leads to divergence of the factors $f1[z]$ and $f2[z]$ if evaluated on the Cauchy Horizon. We could choose T for the point on the Cauchy Horizon and T' for the one elsewhere. The behaviour thus obtained would be clearly of the point separated kernel.

The terms for various orders of κ displayed above diverge as can be seen from the factors of $f1[z]$ and occurring in each one of them in the numerator leading to our main result, the divergence of the noise kernel on the Cauchy Horizon. Factors of $f2[z]$ occur only in cases of point separations different from the one considered above and so do not contribute in this.

It should be noted that the above divergence has been obtained using metric (13). We have been constrained by our approach to use such a metric in coordinates ill behaved at the Cauchy horizon.² Suitable co-ordinate transformations can be used to remedy this. From the metric one can observe that the coordinate transformation factors acting on $N_{TTT'T'}$ above would need to diverge at the Cauchy Horizon as $\frac{1}{z - z_-}$ similar to [5]. The divergence of the expression of the noise kernel above is rather enhanced if the metric is demanded to be regular at the Cauchy horizon.

In the case of covered singularity $(z^2 - R'(z)^2)$ has no real roots. So the factors of $f1[z]$ and $f2[z]$ do not yield divergences. Nor do the rest of the factors. The metric used is also regular everywhere.

The issue of regularity also occurs in the noise expressions of [15] as the Schwarzschild metric used is expressed in the usual coordinates rather than say Kruskal coordinates at the event horizon. For interpreting the expressions on the event horizon, further coordinate transformations would be necessary on the noise kernel.

The above analysis is restricted to the noise kernel with only $\delta T \neq 0$. We obtain a similar divergence on Cauchy Horizon (in the naked singular case) for $\delta z \neq 0$ separation as well. The corresponding noise expressions however are not short enough for explicit display in this communication.

V. RESULTS AND DISCUSSION

We know that the noise kernel affects the induced fluctuations of the metric. In equation (4), however the G^{ret} function required would need a background metric for calculating it. Unfortunately, the largely fluctuating stochastic source as implied by the Noise Kernel for naked singular metric does not allow us to treat the backreaction as negligible. So, our results cannot easily be extended for studying the induced fluctuations on Cauchy Horizon.

At the same time, we note that noise kernel is not very large till near the Cauchy Horizon (in the naked singularity case) and so the background appears to be a good approximation despite backreaction.

Moreover, the stochastic source fluctuating highly near the Cauchy Horizon, could be interpreted for its effect as an *environment* (quantum fields) on the *system* (background metric) [3]. The spacetime could be highly sensitive to decoherence effects very near the Cauchy Horizon. We suggest that this issue could be pursued separately.

The covered singularity case is in contrast with the above results. This is similar to the difference seen in the semiclassical behaviour of the metric. In the latter analysis, the quantum stress tensor diverges on the Cauchy

² The metric in the earlier (t, r, θ, ϕ) co-ordinate system is perfectly regular on Cauchy horizon and we have conformally rescaling back our results. Our results would thus hold for any other metric related to it by regular coordinate system transformations.

horizon in the naked singular case against regular behaviour in the covered case. The contrast is borne out by analysis of fluctuations as well.

Further, the semiclassical interpretation of the stress tensor divergence on Cauchy Horizon [5] would need to be reconsidered in view of our analysis. The highly fluctuating source near the Cauchy horizon would render it difficult to interpret any of its stochastic realizations as physically significant, in particular the semiclassical one of an energy burst on Cauchy Horizon.

The divergence of quantum stress tensor in the semiclassical analysis has been attributed to the fact that high curvature regions are exposed in the naked singular case leading to high energy effects. It would seem that high curvatures lead to diverging noise as well. We suggest that this could be investigated further.

The above interpretations have been suggested using our analysis of the point separated Noise Kernel expressions. Although this point separated form has a physical significance (after smearing with test functions) [15], our results regarding the divergence hold even after a Hadamard subtraction has been carried out on the Wightmann function. This calculation, though not necessary for our interpretations above, underlines the fact that our results are not due to the ultraviolet divergence of the Wightmann function.

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